

Instanton-induced crossover in dense QCD

Naoki Yamamoto

*Department of Physics, The University of Tokyo,
Tokyo 113-0033, Japan
E-mail: yamamoto@nt.phys.s.u-tokyo.ac.jp*

ABSTRACT: We study the properties of an instanton ensemble in three-flavor dense QCD which can be regarded as an instanton plasma weakly interacting by exchanging the η' mesons. Based on this description, we explore the chiral phase transition induced by the instanton ensemble at high baryon density in analogy with the Berezinskii-Kosterlitz-Thouless transition. Using the renormalization group approach, we show that the instanton ensemble always behaves as a screened and unpaired plasma. We also demonstrate that the chiral condensate in dense QCD is proportional to the instanton density.

KEYWORDS: Solitons Monopoles and Instantons, Renormalization Group, QCD.

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1. Introduction

Topological excitations play crucial roles for understanding the properties of various systems in condensed matter physics and particle physics. For example, $O(2)$ spin system in two-dimension is equivalent to a vortex ensemble interacting by two-dimensional Coulomb potential; it shows a second order phase transition from a system composed of vortex dipoles to a vortex plasma as temperature increases. This is known as the Berezinskii-Kosterlitz-Thouless phase transition [1]. Another remarkable example is the three-dimensional compact quantum electrodynamics (QED). It can be described by an equivalent interacting magnetic monopole ensemble and shows a crossover as a function of the coupling constant e [2]. As a result, the area law of the Wilson loop, or the confinement of the fundamental charge, persists for arbitrary value of e .

Moreover, the instanton ensemble have succeeded in illustrating many features of the vacuum of four-dimensional quantum chromodynamics (QCD) and its hadronic observables [3]. Most importantly, it provides a qualitative understanding of the spontaneous breaking of chiral symmetry in the QCD vacuum as well as a possible mechanism of its restoration at finite temperature: numerical calculations in the instanton liquid model show that the chiral restoration corresponds to a transition from an unpaired instanton plasma at low temperature to instanton-antiinstanton molecules at high temperature in the physical case of up, down and strange quarks [4].

Recently, it was shown in the Ginzburg-Landau approach to three-flavor dense QCD [5, 6] that the interplay between the quark-antiquark pairing (chiral condensate) and the quark-quark pairing (diquark condensate) originating from the instanton-induced interaction may lead to a smooth crossover between the hadronic phase and the color superconducting (CSC) phase [7]. If such a crossover is realized, the coexistence phase of the chiral

and diquark condensates extends to the region of high baryon density. However, the dynamical roles of the instanton ensemble in such a system have not been fully studied in the literatures except for a seminal work on the instanton description of two-flavor color superconductivity (2SC) [8]. It was shown in ref. [8] that the low-energy dynamics of two-flavor dense QCD can be described by a nonideal instanton ensemble weakly interacting by exchanging the η mesons due to the fact that the system of instantons is dilute and the $U(1)_A$ symmetry is asymptotically restored at high density. In such a case, the η meson can be regarded as the lightest asymptotic Nambu-Goldstone (NG) boson. By rewriting the low-energy effective Lagrangian of the η meson in the Coulomb gas representation via a duality mapping, two-flavor dense QCD reduces to an instanton ensemble where instantons (antiinstantons) interact with each other by four-dimensional Coulomb potential generated by topological charges.

In the present paper, we will generalize the idea of ref. [8] to three-flavor QCD: We will first provide a complete derivation and its justification of the instanton description of three-flavor dense QCD which was partially suggested but was not fully explored in ref. [8]. Then we will investigate the properties of the instanton ensemble using the renormalization group approach and show that the instanton ensemble behaves as a screened and unpaired plasma. Thus, the chiral condensate inevitably exists even at high baryon density regime. This is consistent with the previous finding in refs. [5, 6] and constitute a dynamical demonstration of the coexistence of the chiral and diquark condensates at high density.

Throughout this paper, we will limit ourselves to three-flavor quark matter with two light degenerate up and down quarks ($m_u = m_d = m_{ud}$) and a medium-heavy strange quark ($m_s > m_{ud}$) at zero temperature and at finite baryon density.¹ We remark here that the light η' meson and the diluteness of instantons enable us to treat the instanton calculations under analytical control at high baryon density: This is not the case in the vacuum and at finite temperature where the assumption of the random instanton liquid needs to be introduced [3].

The paper is organized as follows. In section 2, after describing the instanton ensemble of three-flavor dense QCD, we derive analytical formulas for the instanton density, the topological susceptibility and a dense version of the Witten-Veneziano relation. In section 3, we show that the system of instantons at high baryon density always behave as a screened and unpaired plasma by using the renormalization group approach. Also we illustrate that the chiral condensate induced by the instanton plasma is proportional to the instanton density. Section 4 is devoted to conclusion and summary. In appendix. A, we give the mass spectra of meson excitations at high baryon density.

2. Instanton ensemble at high baryon density

Let us consider how the low-energy dynamics in three-flavor dense QCD can be described by a nonideal instanton ensemble weakly interacting by exchanging the η' mesons. Although the method employed in this section is motivated by the approach proposed in ref. [8],

¹We will not consider another possibility of the exotic state called quarkyonic phase at high baryon density [9].

a complete derivation and its justification for not-fully-explored three-flavor case is given here. First of all, owing to the inverse meson mass ordering, $m_{\eta'} < m_K < m_\pi < m_\eta$, which is caused by the explicit breaking of the flavor SU(3) symmetry ($m_s > m_{ud}$) [10], we can focus on the low-energy effective Lagrangian of the η' meson at high baryon density. This ideal situation has not been realized in the two-flavor case, because only two colors (red and green) participate in the 2SC pairing and there are not only asymptotically massless η meson but unpaired (ungapped) blue quarks.

Our starting point is the three-flavor quark matter where the ground state is the color-flavor locking (CFL) color superconducting phase characterized by diquark condensates [11]:

$$\begin{aligned} \langle q_{Lb}^j C q_{Lc}^k \rangle &= \epsilon_{abc} \epsilon_{ijk} [d_L^\dagger]_{ai}, \\ \langle q_{Rb}^j C q_{Rc}^k \rangle &= \epsilon_{abc} \epsilon_{ijk} [d_R^\dagger]_{ai}. \end{aligned} \quad (2.1)$$

Here i, j, k (a, b, c) are flavor (color) indices and C is the charge conjugation operator. We define the η' meson field ϕ as

$$d_L d_R^\dagger = \left| d_L d_R^\dagger \right| e^{i\phi}. \quad (2.2)$$

The field ϕ transforms as $\phi \rightarrow \phi + 4\alpha_A$ under the $U(1)_A$ rotation $q_L \rightarrow e^{-i\alpha_A} q_L$. The low-energy effective Lagrangian of the η' meson at high density is given by [12–14]:

$$\begin{aligned} \mathcal{L} &= \frac{3}{4} f_{\eta'}^2 [(\partial_0 \phi)^2 - v^2 (\partial_i \phi)^2] - V(\phi), \\ V(\phi) &= -aM \cos(\phi - \theta), \end{aligned} \quad (2.3)$$

where $f_{\eta'}$ is the decay constant of the η' meson and v is the velocity originating from the absence of Lorentz invariance in medium. $V(\phi)$ is the potential induced by one-instanton contribution, $\sim \text{Tr}_{ij} [\hat{M}_{ik} (d_L^\dagger d_R)_{kj}]$ with the quark mass matrix $\hat{M} = \text{diag}(m_u, m_d, m_s)$ and “Tr” is taken over flavor indices. θ is the theta-angle, M is defined as $M = \text{Tr} \hat{M}$ and a is a μ -dependent parameter which we will explicitly calculate below. We neglect the multi-instanton contributions to $V(\phi)$ since they are suppressed due to the diluteness of instantons at high baryon density. It should be remarked that the term $\sim \text{Tr}_{ij} [\hat{M}_{ik} (d_L^\dagger d_R)_{kj}]$ generates not only the mass of the η' meson but also those of other pseudoscalar mesons (π , K and η). The contribution of the $\mathcal{O}(\hat{M}^2)$ -term to eq. (2.3) does not change our discussion basically and is neglected here for simplicity. This will be considered in more detail in appendix A.

At sufficiently large quark chemical potential compared with the typical scale of QCD, $\mu \gg \Lambda_{\text{QCD}}$, $f_{\eta'}$ and v are found by matching to their microscopic values [10]:

$$f_{\eta'}^2 = \frac{3\mu^2}{8\pi^2}, \quad v^2 = \frac{1}{3}. \quad (2.4)$$

In order to obtain the explicit form of $V(\phi)$, let us start with the instanton-induced six-fermion interaction [15, 16, 3]:

$$\begin{aligned} \mathcal{L}_{\text{inst}} = e^{i\theta} \int d\rho n(\rho) \frac{(2\pi\rho)^6 \rho^3}{6N_c(N_c^2 - 1)} \epsilon_{i_1 i_2 i_3} \epsilon_{j_1 j_2 j_3} \left[\frac{2N_c + 1}{2N_c + 4} (\bar{q}_{L i_1} q_{R j_1}) (\bar{q}_{L i_2} q_{R j_2}) (\bar{q}_{L i_3} q_{R j_3}) \right. \\ \left. - \frac{3}{8(N_c + 2)} (\bar{q}_{L i_1} q_{R j_1}) (\bar{q}_{L i_2} \sigma_{\mu\nu} q_{R j_2}) (\bar{q}_{L i_3} \sigma_{\mu\nu} q_{R j_3}) + (L \leftrightarrow R) \right] + \text{h.c.} \end{aligned} \quad (2.5)$$

Here ρ is the instanton size, N_c is the number of colors, $i_{1,2,3}$ and $j_{1,2,3}$ are flavor indices and $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$. The instanton size distribution $n(\rho)$ is given by [17, 3]

$$n(\rho) = C_N \left(\frac{8\pi^2}{g^2} \right)^{2N_c} \rho^{-5} \exp\left(-\frac{8\pi^2}{g(\rho)^2}\right) e^{-N_f \mu^2 \rho^2}, \quad (2.6)$$

$$C_N = \frac{0.466 \exp(-1.679N_c) 1.34^{N_f}}{(N_c - 1)!(N_c - 2)!}, \quad (2.7)$$

$$\frac{8\pi^2}{g(\rho)^2} = -b \log(\rho \Lambda_{\text{QCD}}), \quad b = \frac{11}{3}N_c - \frac{2}{3}N_f, \quad (2.8)$$

where N_f is the number of flavors. Replacing one of $\bar{q}_L q_R$ with \hat{M} in eq. (2.5) and taking the expectation value with respect to the CFL ground state (2.1), where

$$|d_L| = |d_R| = \sqrt{\frac{6N_c}{N_c + 1} \frac{\mu^2 \Delta}{\pi g}}, \quad (2.9)$$

with Δ being the superconducting gap near the Fermi surface, one finds [12–14]:

$$V(\phi) = - \int d\rho n(\rho) \frac{2(2\pi\rho)^4 \rho^3}{N_c(N_c - 1)} |d_L|^2 2M \cos(\phi - \theta). \quad (2.10)$$

The integration over the instanton size ρ above results in the form of eq. (2.3), where the coefficient a is given by

$$a(\mu) = \frac{24}{N_c^2 - 1} C_N N_f^{-\frac{b+3}{2}} \Gamma\left(\frac{b+3}{2}\right) \left(\frac{8\pi^2}{g^2}\right)^{2N_c+1} \left(\frac{\Lambda_{\text{QCD}}}{\mu}\right)^b \mu \Delta^2, \quad (2.11)$$

with $\Gamma(x)$ being the gamma function. The well-known infrared divergence in instanton calculation in the QCD vacuum is not seen here since the instanton screening factor, $e^{-N_f \mu^2 \rho^2}$ in eq. (2.6) [17], gives the small size $\rho \sim \mu^{-1}$ of instantons and regulate the integral.

By rescaling $\phi \rightarrow 2\phi/(\sqrt{3}v f_{\eta'})$ and using the new coordinate $x_0 = v\tau$ with the imaginary time τ , the effective action of η' in eq. (2.3) reduces to the Euclidean invariant form:

$$S_E = \int d^4x [(\partial\phi)^2 - \lambda \cos \alpha(\phi - \theta)], \quad (2.12)$$

$$\lambda = \frac{a}{v} M, \quad \alpha = \frac{2}{\sqrt{3}v f_{\eta'}}. \quad (2.13)$$

We note that the parameter α is a function of chemical potential since $f_{\eta'} \sim \mu$. The instanton potential gives the η' mass as

$$m_{\eta'}^2 = \frac{\lambda}{2} \alpha^2 = \frac{16\pi^2 a}{3\mu^2} M, \quad (2.14)$$

where the second equation holds from the weak coupling relation, eq. (2.4). Therefore, $a \rightarrow 0$ as $\mu \rightarrow \infty$ from eq. (2.11) and the η' meson is a NG boson at high baryon density limit.

Via a dual transformation, the partition function for the action in eq. (2.12) reduces to the following form [8]:

$$\begin{aligned} Z &= \int \mathcal{D}\phi e^{-S_E} = \int \mathcal{D}\phi e^{-\int d^4x (\partial\phi)^2} e^{\lambda \int d^4x \cos \alpha(\phi(x)-\theta)} \\ &= \sum_{N_{\pm}=0}^{\infty} \frac{(\lambda/2)^N}{N_+!N_-!} \int d^4x_1 \dots \int d^4x_N \int \mathcal{D}\phi e^{-\int d^4x (\partial\phi)^2} e^{i \sum_{i=0}^N Q_i \alpha(\phi(x_i)-\theta)}, \end{aligned} \quad (2.15)$$

where the sum is taken over possible sets of N_+ (N_-) with positive (negative) charge $Q_i = \pm 1$ located at the position x_i . In deriving the second line in eq. (2.15), we have used the relation,

$$\lambda \cos \alpha(\phi(x) - \theta) = \frac{\lambda}{2} \sum_{Q=\pm 1} e^{iQ\alpha(\phi(x)-\theta)}. \quad (2.16)$$

Integrating over the variable $\phi(x)$ in eq. (2.15), one ends up with [8]

$$Z = \sum_{N_{\pm}=0}^{\infty} \frac{(\lambda/2)^N}{N_+!N_-!} \int d^4x_1 \dots \int d^4x_N e^{-i\theta \sum_{i=0}^N Q_i} e^{-\sum_{i>j=0}^N Q_i Q_j G(x_i - x_j)}, \quad (2.17)$$

which is a Coulomb gas representation of the original sine-Gordon model. Since the θ -angle is conjugate to the topological charge in QCD, $Q = \sum_i Q_i = N_+ - N_-$ is identified with the total topological charge and N_+ (N_-) with the number of instantons (antiinstantons). Also,

$$G(x_i - x_j) = \frac{\alpha^2}{8\pi^2(x_i - x_j)^2} \quad (2.18)$$

is the four-dimensional Coulomb potential between instantons (antiinstantons). Therefore, eq. (2.17) exhibits that this system is an instanton ensemble in which instantons and antiinstantons with topological charge $Q_i = \pm 1$ interact with each other by the potential $G(x_i - x_j)$.

Note that we can treat our instanton calculations under completely analytical control depending on two distinctive facts in dense QCD:

- (i) Instantons are sufficiently dilute indicated by the parameter $\Lambda_{\text{QCD}}/\mu \ll 1$, which enables us to deal with the effects of instantons as a perturbation.
- (ii) The inverse mass ordering of pseudoscalar mesons, $m_{\eta'} < m_K < m_{\pi} < m_{\eta}$ [10], guarantees that the low-energy dynamics is dominated by the η' mesons.² This is the characteristics with three-flavor and can be confirmed at sufficiently large baryon density (See eqs. (A.1)–(A.7) in appendix A).

More quantitative estimate on the domain of applicability of this instanton description will be discussed in section 4.

²We neglect the exact massless H boson associated with the breaking of the $U(1)_B$ symmetry since its dynamics is totally independent here and decouples from the low-energy effective Lagrangian of η' .

2.1 Instanton density and topological susceptibility

In this subsection, we calculate quantities based on the instanton ensemble discussed above.

Multiplying N_+ (N_-) in the right hand side of eq. (2.15), one finds the expectation value of the instanton (antiinstanton) number as

$$\langle N_+ \rangle = \langle N_- \rangle = \frac{\lambda v}{2} V_4, \quad (2.19)$$

with four-volume $V_4 = \int d\tau d^3x = \int dx_0 d^3x/v$. This shows that the average of the topological charge $\langle Q \rangle = \langle N_+ \rangle - \langle N_- \rangle$ vanishes and the instanton density as defined below reads

$$n_{\text{inst}} = \frac{\langle N \rangle}{V_4} = \lambda v \quad (2.20)$$

with $N = N_+ + N_-$. This result has been obtained in ref. [8].

Moreover, by using eq. (2.15), we generally obtain the mixed factorial moments:

$$\left\langle \frac{N_+!}{(N_+ - k)!} \frac{N_-!}{(N_- - l)!} \right\rangle = \left(\frac{\lambda v}{2} V_4 \right)^{k+l}, \quad (2.21)$$

for arbitrary nonnegative integers k and l . eq. (2.21) implies that instantons and antiinstantons independently follow the Poisson distribution,

$$f(x) = e^{-\beta} \frac{\beta^x}{x!}, \quad (2.22)$$

with $\beta = \lambda v V_4/2 = \langle N \rangle/2$, from the fact that the n -th factorial moment of the Poisson distribution is equal to β^n . This Poissonian behavior is usually assumed in the QCD vacuum [3], but it can be justified at high baryon density for a dilute system of interacting instantons and antiinstantons as anticipated. Also, for the topological susceptibility defined by

$$\chi_{\text{top}} = \frac{\langle Q^2 \rangle}{V_4}, \quad (2.23)$$

we have a simple relation,

$$\chi_{\text{top}} = n_{\text{inst}} = \lambda v, \quad (2.24)$$

as a property of the Poisson distribution. By the use of eq. (2.24), the η' mass in eq. (2.14) reduces to³

$$m_{\eta'}^2 = \frac{2\chi_{\text{top}}}{3f_{\eta'}^2 v^2}. \quad (2.25)$$

This is a dense version of the Witten-Veneziano relation [18] obtained as a natural application of the instanton ensemble, which is not given in ref. [8].

³The topological susceptibility in eq. (2.24) and the Witten-Veneziano relation (2.25) are consistent with the results of ref. [14] at high baryon density where the two-instanton term is negligible, though the factor v^2 in eq. (2.25) does not appear in [14]. This difference comes from the fact that our η' mass is defined to satisfy the dispersion relation $E^2 = v^2(\mathbf{p}^2 + m_{\eta'}^2)$ while that in ref. [14] is the pole mass, i.e., the energy of η' at $\mathbf{p} = \mathbf{0}$, $m_{\eta'}^{(\text{pole})} = vm_{\eta'}$.

3. Renormalization group analysis on instanton ensemble

In this section, we consider the possible phases of instantons at high baryon density on the basis of the instanton description in section 2. In the following, we set $\theta = 0$ for simplicity since all the arguments are independent of the parameter θ . In order to explore and compare the general properties of phase transitions induced by the D -dimensional topological excitations ($D = 2$ for vortices, $D = 3$ for monopoles and $D = 4$ for instantons), we generalize eq. (2.12) to the D -dimensional sine-Gordon model whose action is given by

$$S_D = \int d^D x [(\partial\phi)^2 - \lambda_D \cos \alpha\phi]. \quad (3.1)$$

Here α is the parameter with the mass dimension $1 - D/2$, which is introduced after appropriate rescaling the field ϕ so that we normalize the coefficient of the kinetic term to be 1.

The long-range Coulomb force between topological excitations requires effects of many-body dynamics or quantum fluctuations. For this purpose, we shall now perform the Wilson renormalization group (RG) approach and divide $\phi(x)$ into two components, $\phi = \phi' + \delta\phi$ with low-momentum part $0 < k < \Lambda'$ and high-momentum part $\Lambda' < k < \Lambda$ respectively, where Λ' is smaller than Λ by an exponential factor. This RG analysis for $D = 2$ has been already carried out in ref. [1], and we extend it to the case of $D \geq 3$ in the following. Considering how the small coupling $\lambda \ll 1$ with the predominantly Gaussian fluctuations shifts after the RG transformation, the change of the potential term can be calculated by integrating out the momentum shell $\Lambda' < k < \Lambda$ as

$$\langle \cos \alpha(\phi' + \delta\phi) \rangle = \frac{1}{2} \left(e^{i\alpha\phi'} e^{-\alpha^2 \langle \delta\phi^2 \rangle_{D/2}} + \text{c.c.} \right), \quad (3.2)$$

with

$$\langle \delta\phi^2 \rangle_D = \sum_{\vec{k}} \frac{1}{k^2} = \int \frac{d\Omega_D}{(2\pi)^D} \int_{\Lambda'}^{\Lambda} \frac{k^{D-1}}{k^2} dk, \quad (3.3)$$

where Ω_D is the surface area of a unit sphere in Euclidean D -dimension. Therefore, the form of sine-Gordon action, eq. (3.1), is preserved and changed to

$$S_D \rightarrow S_{D'} = \int d^D x ((\partial\phi')^2 - \lambda_D^* \cos \alpha\phi'), \quad (3.4)$$

with the coupling constant

$$\begin{aligned} \lambda_{D=2}^* &= x^{\alpha^2/4\pi} \lambda_D, \\ \lambda_{D \geq 3}^* &= \exp \left[\frac{\alpha^2 \Lambda^{D-2} (x^{D-2} - 1)}{(D-2) 2^D \pi^{D/2} \Gamma(D/2)} \right] \lambda_D, \end{aligned} \quad (3.5)$$

for $D = 2$ and $D \geq 3$ respectively. Here we define the renormalization scale $x = \Lambda'/\Lambda < 1$. At the same time, the kinetic term $(\partial\phi')^2$ is effectively reduced by the factor of $x^2 < 1$ independent of the dimension D , since ∂_μ is of order Λ and ϕ' is of order Λ' .

	D	topological excitation	parameter α	order of phase transition
(a)	2	vortex	$\alpha \propto \sqrt{J/T}$	second order
(b)	3	magnetic monopole	$\alpha \propto 1/e$	crossover
(c)	4	instanton	$\alpha = 2/(\sqrt{3}vf_{\eta'}) \sim 1/\mu$	crossover

Table 1: Order of phase transitions of (a) two-dimensional $O(2)$ spin model with the nearest neighboring interaction J [1], (b) three-dimensional compact QED with the coupling constant e [2] and (c) four-dimensional dense QCD with quark chemical potential μ . In each case, D -dimensional ($D = 2, 3, 4$) sine-Gordon model is equivalent to an ensemble of Coulomb-like interacting topological excitations. Parameters α to exhibit phase transitions are also shown.

The systems described by the D -dimensional ($D = 2, 3, 4$) sine-Gordon model are summarized as follows: (a) two-dimensional $O(2)$ spin model with the nearest neighboring interaction J [1], (b) three-dimensional compact QED with the coupling constant e [2], and (c) four-dimensional dense QCD with quark chemical potential μ . They are respectively equivalent to an ensemble of vortices, magnetic monopoles and instantons interacting by the D -dimensional Coulomb potential. The resultant orders of phase transitions are summarized in table. 1. The parameter α in each case is also given.

3.1 Case (a): two-dimensional $O(2)$ spin model

As a pedagogical demonstration, we first recall the case (a) and consider which is overwhelming after the RG transformation, the potential term or the kinetic term in accordance with ref. [1]. From eq. (3.5), when $\alpha^2/4\pi > 2$, the potential term is suppressed by fluctuations so quickly that it is irrelevant compared to the kinetic term. Therefore, the system can be described only by the spin wave in this case. In the language of the Coulomb gas representation, this corresponds to an insulating phase where vortex and antivortex occur in pairs. Otherwise, i.e., $\alpha^2/4\pi < 2$, the potential term takes over the kinetic term regardless of the initial value of λ and the system is locked in one of the cosine minima $\phi = 2\pi n/\alpha$ with integer n . This corresponds to a plasma phase where the Coulomb potential is screened by the free vortices. As a result, the system changes from vortex dipoles to a vortex plasma on reaching $\alpha^2/8\pi = 1$ as temperature increases. Also we can easily check that the transition temperature $T_c = J/8\pi$ is identical to the prediction obtained from the interplay between the free-energy and the entropy of the vortex ensemble [1].

3.2 Case (b): three-dimensional compact QED

For $D \geq 3$, on the other hand, the kinetic term $\sim x^2$ vanishes while the potential term λ remains finite in the limit $x \rightarrow 0$, unlike λ also vanishes for $D = 2$. This originates from the fact that the integral in eq. (3.3) is infrared divergent only for $D = 2$, but is finite for $D \geq 3$. Therefore, the kinetic term is more suppressed than the potential after the RG transformation and topological excitations for $D \geq 3$ always behave as a screened and unpaired plasma.

As a result, in the case (b), the magnetic monopoles resides in a screened plasma phase and show a crossover as a function of the coupling constant e . Since the area law of the

Wilson loop can be proven for the strong coupling limit $e \gg 1$, it leads to a well-known conclusion that the confinement of the fundamental charge persists for arbitrary value of e in the three-dimensional compact QED, which was first shown in ref. [2].

3.3 Case (c): four-dimensional QCD at finite baryon density

Let us now turn back to the pending question of our interest, whether the system of instantons acts as an instanton plasma or they couple into molecules in the case (c). In an analogous fashion to the previous subsection, we find that the system of instantons always behaves as a screened and unpaired plasma and shows a crossover as a function of $f_{\eta'} \sim \mu$. Since unpaired instantons induce the formation of quark-antiquark pairing and give nonvanishing chiral condensate, our result implies that the chiral condensate will remain finite in the region of high baryon density.⁴

More quantitatively, we can calculate the chiral condensate in relation to our instanton ensemble. The minimum of the potential $V(\phi)$ in eq. (2.3) is given at $\phi = \theta$:

$$V(\phi)_{\min} = -aM. \tag{3.6}$$

Differentiating this energy with respect to M , one obtains the chiral condensate as

$$\langle \bar{q}q \rangle_{\text{csc}} = -a = -\frac{n_{\text{inst}}}{M}, \tag{3.7}$$

where we have used eqs. (2.13) and (2.24). eq. (3.7) is a novel relation connecting the chiral condensate to the instanton density in dense QCD. Since the instanton density rapidly decreases at high baryon density like $n_{\text{inst}} \propto \lambda \sim \mu^{1-b}$ with $b = 9$ for $N_c = N_f = 3$ from eqs. (2.8) and (2.11), the chiral condensate is highly suppressed (but remains finite) like $\langle \bar{q}q \rangle_{\text{csc}} \sim \mu^{1-b}$.

This is a remarkable consequence, since previous studies using three-flavor effective model calculations such as the Nambu-Jona-Lasinio (NJL) model [20] and the random matrix model [21], exhibit the pure CSC phase without the chiral condensate is realized at high baryon density. This difference comes from the fact that they neglect the effects of instantons in the CFL ground state, which would be a trigger of the chiral condensate. Actually, the coexistence phase of the chiral and diquark condensates at high baryon density has been recently reported based on the model-independent Ginzburg-Landau approach taking into account the instanton effects properly [5, 6]. The important point there is that the instanton-induced interaction composed of the chiral and diquark condensates:

$$\mathcal{L}_{\text{ext}} = \gamma \text{Tr}[(d_R d_L^\dagger)(\bar{q}_R q_L) + \text{h.c.}], \tag{3.8}$$

acts an external field for the chiral condensate and leads to a chirally broken crossover between the hadronic phase and the CSC phase. Our result of the coexistence phase

⁴The application of our argument here to two-flavor QCD is not straightforward, since there are not only light η mesons but nearly massless unpaired blue quarks in the 2SC phase, as mentioned in section 2. However, the instanton liquid model with two-flavor shows a tendency towards chiral restoration by forming instanton molecules at high baryon density [19].

at high baryon density is totally consistent with this observation due to the same origin of instantons.

It should be remarked that the chiral condensate in dense QCD is proportional to the instanton density in eq. (3.7), which is in contrast with the case of the QCD vacuum with $N_f \geq 2$ where the chiral condensate is expected to behave as [3]

$$\langle \bar{q}q \rangle_{\text{vac}} \propto -\frac{n_{\text{inst}}^{1/2}}{\rho}. \quad (3.9)$$

This difference can be understood as follows: The spontaneous breaking of chiral symmetry in the QCD vacuum is a collective phenomena caused by the effect of infinitely many instantons, and the chiral condensate must be determined from self-consistent relations, which finally results in eq. (3.9) [3]. On the other hand, in the case of dense QCD, chiral symmetry is broken by a single instanton effect thanks to the presence of diquarks as shown in eq. (3.8), and it is anticipated that the chiral condensate is proportional to the number of instantons N in a four-volume V_4 , i.e., the instanton density n_{inst} .

4. Discussion and conclusion

In this paper, we have studied the properties of an instanton ensemble in three-flavor dense QCD which can be regarded as an instanton plasma weakly interacting by exchanging the η' mesons. Based on this description, we derive analytical formulas for the instanton density, the topological susceptibility and a dense version of the Witten-Veneziano relation. We also explore the chiral phase transition induced by the instanton ensemble in analogy with the Berezinskii-Kosterlitz-Thouless transition. We generally show that the system of Coulomb interacting D -dimensional topological excitations exhibits a second order phase transition for $D = 2$, and a crossover for $D \geq 3$ using the renormalization group approach. In particular, for $D = 4$, the instanton ensemble always behaves as a screened and unpaired plasma, which gives nonvanishing chiral condensate proportional to the instanton density at high baryon density regime of QCD. Therefore, the coexistence phase of the chiral and diquark condensates is inevitably expected in dense QCD as suggested in refs. [5, 6].

The discussion on the applicable domain of the instanton description introduced above is in order here. Our treatment is based on the low-energy effective Lagrangian of the η' meson, eq. (2.3), which is valid when two conditions on the η' pole mass are satisfied: (i) $m_{\eta'} \lesssim 2\Delta$, and (ii) $m_{\eta'} \lesssim m_{\pi,K,\eta}$. The condition (i) is required since, otherwise ($m_{\eta'} > 2\Delta$), η' would rapidly decay into a particle-hole pair and becomes unstable. Also the condition (ii) is necessary to assure that we have only to focus on the low-energy effective Lagrangian of the light η' meson. When $\mu \gg \Lambda_{\text{QCD}}$, $m_{\eta'} \ll 2\Delta$ as well as the inverse meson mass ordering, $m_{\eta'} < m_K < m_\pi < m_\eta$ follows due to $a \ll 1$, so that the conditions (i) and (ii) are satisfied (See appendix A). Moreover, we find the critical chemical potential μ_c as $\mu_c \sim 10\Lambda_{\text{QCD}}$ for $m_{ud} = 5\text{-}10 \text{ MeV}$, $m_s = 150 \text{ MeV}$ and $\Lambda_{\text{QCD}} = 200 \text{ MeV}$.

The extrapolation of the instanton-induced crossover obtained here to lower baryon density is a nontrivial question which we cannot address within our treatment. However, it might be reasonable to expect that the system of instantons behaves as a gas-like

weakly-correlated or a liquid-like strongly-correlated plasma across the entire span of the density. The instanton-induced crossover may have relevance to the continuity between hadronic phase and color superconductivity phase [23, 24] and the spectral continuity of hadrons [6, 25] from low to high baryon densities. It would be also important to study how the confinement-deconfinement phase transition at finite baryon density is related to the changes in the behavior of an instanton ensemble [26].

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A. Mass spectra of meson excitations

The explicit inclusion of the $\mathcal{O}(M^2)$ -term does not change our discussion in a substantial way. But it is rather essential to validate the low-energy effective Lagrangian of the η' meson, eq. (2.3), so that we can neglect other heavier meson excitations. In the case of two light degenerate up and down quarks with a medium-heavy strange quark, flavor SU(2) symmetry is respected but flavor SU(3) symmetry is not. Then we find the masses of pions (π^0 and π^\pm) and kaons (K^\pm , K^0 and \bar{K}^0) to the order of $\mathcal{O}(M^2)$ as [10, 12, 28, 29]:

$$m_{\pi^\pm, \pi^0} = \left[\frac{2a}{f_\pi^2} m_{ud} + \frac{8C}{f_\pi^2} m_{ud} m_s \right]^{1/2}, \quad (\text{A.1})$$

$$m_{K^\pm, K^0, \bar{K}^0} = \mp \frac{m_s^2 - m_{ud}^2}{2\mu} + \left[\frac{a}{f_\pi^2} (m_{ud} + m_s) + \frac{4C}{f_\pi^2} m_{ud} (m_s + m_{ud}) \right]^{1/2}, \quad (\text{A.2})$$

where the first term on the right hand side of eq. (A.2) is the effective modifications of the chemical potential due to the explicit breaking of the flavor SU(3) symmetry [27]. The coefficient C and the pion decay constant f_π have been determined from weak-coupling calculations at high density [10]:

$$C = \frac{3\Delta^2}{4\pi^2}, \quad (\text{A.3})$$

$$f_\pi^2 = \frac{21 - 8 \ln 2}{18} \frac{\mu^2}{2\pi^2}. \quad (\text{A.4})$$

On the other hand, the neutral mesons, η and η' , are unaffected by the effective chemical potential. However, since η' mixes with η , the diagonalization of the 2×2 mass matrix

m_{ab}^2 ($a, b = 0, 8$),

$$\begin{aligned} m_{00}^2 &= \frac{8C}{3f_{\eta'}^2} m_{ud}(2m_s + m_{ud}), \\ m_{08}^2 &= \frac{8\sqrt{2}C}{3f_{\eta'}f_{\pi}} m_{ud}(m_s - m_{ud}), \\ m_{88}^2 &= \frac{8C}{3f_{\pi}^2} m_{ud}(m_s + 2m_{ud}), \end{aligned} \tag{A.5}$$

is necessary to obtain the genuine mass eigenvalues of η' . Also taking into account the instanton contribution to η' , eq. (2.14), their masses finally turn out to be

$$m_{\eta'} = \left[\frac{2a}{3f_{\eta'}^2} M + \frac{24C}{2f_{\pi}^2 + f_{\eta'}^2} m_{ud}^2 \right]^{1/2}, \tag{A.6}$$

$$\begin{aligned} m_{\eta} = & \left[\frac{a}{3f_{\pi}^2} (m_u + m_d + 4m_s) + \left(\frac{1}{f_{\pi}^2} + \frac{2}{f_{\eta'}^2} \right) \frac{8C}{3} m_{ud} m_s \right. \\ & \left. + \left(\frac{2}{f_{\pi}^2} + \frac{1}{f_{\eta'}^2} - \frac{9}{2f_{\pi}^2 + f_{\eta'}^2} \right) \frac{8C}{3} m_{ud}^2 \right]^{1/2}. \end{aligned} \tag{A.7}$$

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